

Approximation Techniques for Estimating Parameters of a Stochastic Sediment Transport Model

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Abstract

Approximation techniques for estimating stochastic parameters of the non-homogeneous Poisson (NHP) model are presented. The NHP model is characterized by a two-parameter cumulative probability distribution function (CDF) of sediment displacement. These two parameters are, respectively, the temporal and spatial intensity functions physically representing the inverse of average rest period and step length of sediment particles. The NHP model is mathematically an improvement over the earlier homogeneous model initiated by Einstein in 1937, however, difficulty of estimating parameters has set limitations on its applications. The approximation techniques are proposed to address such problem.

The basic idea of the method is to approximate a model involving stochastic parameters by Taylor series expansion which preserving certain higher-order terms of interest. With the input of physically measured data, model parameters can be determined through a system of equations simplified by the approximation technique. The parameters so determined are used to predict the cumulative probability of sediment distribution.

Flume studies of sediment transport in gravel bed are carried out for verification. The computation results obtained with the first- and second-order approximated parameters are compared. The second-order results explicitly reveal a better coincidence with the physical data. The average gross error of the results gained from second-order approximation method is 47% less than that of first-order results. Analyses on the error of estimated parameters and the computation error resulting from the use of an approximate parameter are also performed.

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1 Introduction

Stochastic processes have been applied to modeling sediment transport for over six decades. Einstein [1] presented, perhaps, the first stochastic model of sediment transport based on the concept that bed-load particles move in a sequence of alternate steps and rests. Shen and Todorovic [2], eliminating certain idealized assumptions used by Einstein, developed a general stochastic model for 1-D movement of bed material. The non-homogeneous Poisson (NHP) model of Shen-Todorovic is described by the cumulative probability distribution function (CDF) of sediment displacement:

$$\begin{aligned}
 F_t(x) &= \text{Prob}(X_t \leq x) \\
 &= \exp\left[-\int_{t_0}^t \lambda_1(\tau) d\tau\right] \cdot \exp\left[-\int_{x_0}^x \lambda_2(\xi) d\xi\right] \cdot \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \frac{\left[\int_{t_0}^t \lambda_1(\tau) d\tau\right]^n}{n!} \frac{\left[\int_{x_0}^x \lambda_2(\xi) d\xi\right]^j}{j!} \\
 &= \exp[-\Lambda_1(t)] \cdot \exp[-\Lambda_2(x)] \cdot \sum_{n=0}^{\infty} \sum_{j=n}^{\infty} \frac{[\Lambda_1(t)]^n}{n!} \frac{[\Lambda_2(x)]^j}{j!} \quad (1)
 \end{aligned}$$

in which X_t denotes x-direction displacement of a particle at time t ; λ_1 and λ_2 are temporal and spatial intensity functions physically representing the inverse of average rest period and step length; Λ_1 and Λ_2 are the corresponding integral intensity functions. Mathematically the NHP model is an improvement over the earlier one. However, complexity of the model and difficulty in determining parameters has limited its utilities.

This work presents first- and second-order approximation techniques for estimating stochastic parameter of the NHP model. The basic idea of such method is to approximate a model involving stochastic parameters by Taylor series expansion. Laboratory flume studies are carried out for verification. The results gained from the first- and second-order methods are compared, and an analysis on the computation error of the approximation method is also performed.

2 Approximation Techniques

Wu and Shen [3] proposed a first-order approximation technique to approach the temporal and spatial intensity functions, λ_1 and λ_2 , of the NHP model. An alternative method to evaluate the integral temporal intensity function, Λ_1 , was also presented. It was claimed that $\Lambda_1(t)$ is the term that actually governs the temporal portion of Eq. (1) when the spatial distribution of sediment at a specified time t is to be predicted. Hence the alternative approach for determining Λ_1 was recommended. Nevertheless, at this stage λ_2 must be evaluated with Taylor-series approximation. This paper is to present approximation techniques for estimating the spatial intensity function.

To develop the approximation technique for estimating parameters, the CDF of sediment displacement is expanded both forwards and backwards with respect to a selected spatial point, x , by finite increments, Δx and $2\Delta x$. This is accomplished by introducing the Taylor series of the integral spatial intensity function, which preserving certain higher-order terms of interest, into the forward- and backward-expansions of (1). Dividing the expanded forms of (1) by the original CDF leads to further simplified forms. The simplified forward- and backward-expansions are solved as a system to evaluate the parameter λ_2 . The systems of equations derived by first- and second-order methods [4] are listed in Table 1, in which a_1 , a_2 , b_1 , and b_2 , all varying as functions of x , are the unknowns to be solved. $E_i^{t_0, t}$ is the event of making exactly i steps within the period $[t_0, t]$; X_i is the distance traveled after making i steps. Magnitudes of $F_t(x - 2\Delta x)$ through $F_t(x + 2\Delta x)$ are determined from the data measured at a specific time t . Since that $a_1 = \Lambda'_2 \cdot \Delta x$ and $\Lambda'_2 = \lambda_2$, the magnitude of λ_2 at the specified location x can be obtained once a_1 is solved. The spatial intensity function is pursued by solving the system of equations incrementally with respect to the locations where physical data are gathered.

Table 1: Systems of Simplified CDF Equations Used for Parameter Estimation

| | |
|---|---|
| 1st-order method | $\frac{F_t(x + \Delta x)}{F_t(x)} = \exp(-a_1) \cdot (1 + a_1 \cdot b_1)$ $\frac{F_t(x - \Delta x)}{F_t(x)} = \exp(a_1) \cdot (1 - a_1 \cdot b_1)$ |
| 2nd-order method | $\frac{F_t(x + \Delta x)}{F_t(x)} = \exp(-a_1 - a_2) \cdot \left[1 + (a_1 + a_2) \cdot b_1 + \frac{1}{2} a_1^2 \cdot b_2 \right]$ $\frac{F_t(x - \Delta x)}{F_t(x)} = \exp(a_1 - a_2) \cdot \left[1 + (-a_1 + a_2) \cdot b_1 + \frac{1}{2} a_1^2 \cdot b_2 \right]$ $\frac{F_t(x + 2\Delta x)}{F_t(x)} = \exp(-2a_1 - 4a_2) \cdot \left[1 + (2a_1 + 4a_2) \cdot b_1 + 2a_1^2 \cdot b_2 \right]$ $\frac{F_t(x - 2\Delta x)}{F_t(x)} = \exp(2a_1 - 4a_2) \cdot \left[1 + (-2a_1 + 4a_2) \cdot b_1 + 2a_1^2 \cdot b_2 \right]$ |
| $a_1 = (\Delta x) \cdot \frac{F_t'(x)}{F_t(x)}$ $a_2 = (\Delta x^2) \cdot \frac{F_t''(x)}{2F_t(x)}$ | $b_1 = \left[\frac{P(E_0^{t_0, t})}{P(X_t \leq x)} + \frac{\sum_{n=1}^{\infty} P(X_{n-1} \leq x) P(E_n^{t_0, t})}{P(X_t \leq x)} \right]$ $b_2 = \left[\frac{P(E_0^{t_0, t})}{P(X_t \leq x)} + \frac{P(E_1^{t_0, t})}{P(X_t \leq x)} + \frac{\sum_{n=2}^{\infty} P(X_{n-2} \leq x) P(E_n^{t_0, t})}{P(X_t \leq x)} \right]$ |

3 Experiments

Sediment transport and resulting deposition in gravel-bed rivers has been an issue of considerable concern in many natural processes. Modeling the spatiotemporal distribution of sediment in such environment is essential for assessing the consequential impacts. The aim of this experimental study is mainly to verify the proposed parameter estimation techniques and grasp a better understanding of the physical process. Experiments are conducted in a tilting flume of 20cm×40cm cross-section paved with 5-cm-thick gravel substrate, as shown in Figure 1. A sand strip of a predetermined amount is placed across the section at an upstream point of the gravel bed and subjected to a steady flow. For each trial, flow is terminated and water is drained out after a certain running period. Samples are then taken longitudinally along the flume for every 8cm of gravel matrix (indicated as Δx in Fig. 1). The quantities of sediment remained at source area and within the gravel substrate (respectively m_0 and m_1 through m_7 in Fig. 1) are physically measured. Quantitative values of the cumulative probability of particle distribution can be determined from the measured data, i.e.

$$F_t(x_n) = \frac{\sum_{i=0}^n m_i}{M_T} \quad (2)$$

where $x_n = n \cdot \Delta x$ for $n = 0, 1, 2, \dots, 7$; M_T is the quantity of sand initially introduced as upstream source. Once the cumulative probabilities at various locations are evaluated, they can be substituted into the system of simplified equations to seek solutions.

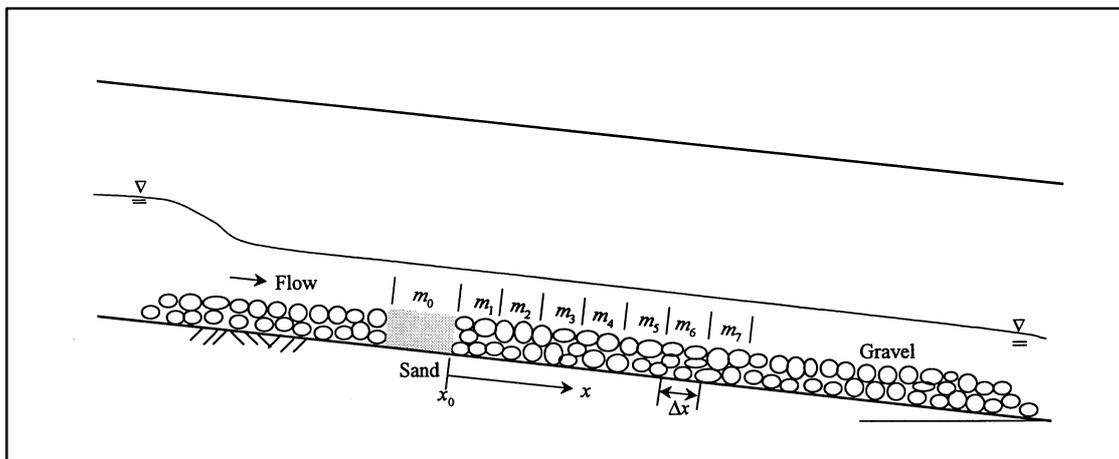


Figure 1: Schematic Diagram of Experimental Setup and Configuration

4 Results

4.1 Spatial Intensity Function, λ_2

The magnitudes of λ_2 at incremental distances from the sand strip are calculated through the procedures of first- and second-order approximation techniques. Fig. 2 shows these data points of calculated λ_2 and their fitting curves that can be represented by a general form:

$$\lambda_2(x) = a \cdot e^{-bx} \quad (3)$$

in which a and b are fitting coefficients. Fig. 2 reveals that first- and second-order λ_2 curves have nearly identical slope on the semi-logarithmic graph, and the magnitude of first-order λ_2 curve is greater than that of the second-order curve. The monotonously descending trend of λ_2 curve is due to the increasing void space available for sediment particles to move through.

4.2 Cumulative Probability Distribution of Sediment Displacement

Respectively using the first- and second-order approximated spatial intensity functions as the parameter, the cumulative probability distributions of sediment displacement can be calculated with Eq. (1). The computed cumulative probability distributions and the experimental data are shown in Fig. 3. The results obtained by employing second-order λ_2 curves explicitly reveal better coincidences with the experimental data. The magnitudes of Euclidean norm, a measure of gross error between the computational and physical data, for all experimental runs are summarized in Table 2. The gross errors of the second-order results are consistently smaller. The percentage of reduced gross error ranges from 12% to 61%, with an average of 47%.

Table 2: Comparison of computation error

| Experiment | Euclidean norm $\ e\ _2$ | | Percentage of error reduced [(1)-(2)]/(1) | Average percentage of error reduced |
|------------------|--------------------------|------------------|--|-------------------------------------|
| | 1st-order (1) | 2nd-order (2) | | |
| LC-1-FL2-S20-30S | 0.032 | 0.018 | 43% | |
| LC-1-FL2-S20-1M | 0.050 | 0.024 | 53% | |
| LC-1-FL2-S20-10M | 0.057 | 0.029 | 49% | 49% |
| LC-1-FL3-S13-30S | 0.034 | 0.014 | 60% | |
| LC-1-FL3-S13-5M | 0.042 | 0.016 | 61% | |
| LC-1-FL3-S13-30M | 0.049 | 0.021 | 58% | 59% |
| LF-1-FL25-S6-5M | 0.043 | 0.038 | 12% | |
| LF-1-FL25-S6-30M | 0.057 | 0.034 | 41% | 27% |
| | | | | 47% |

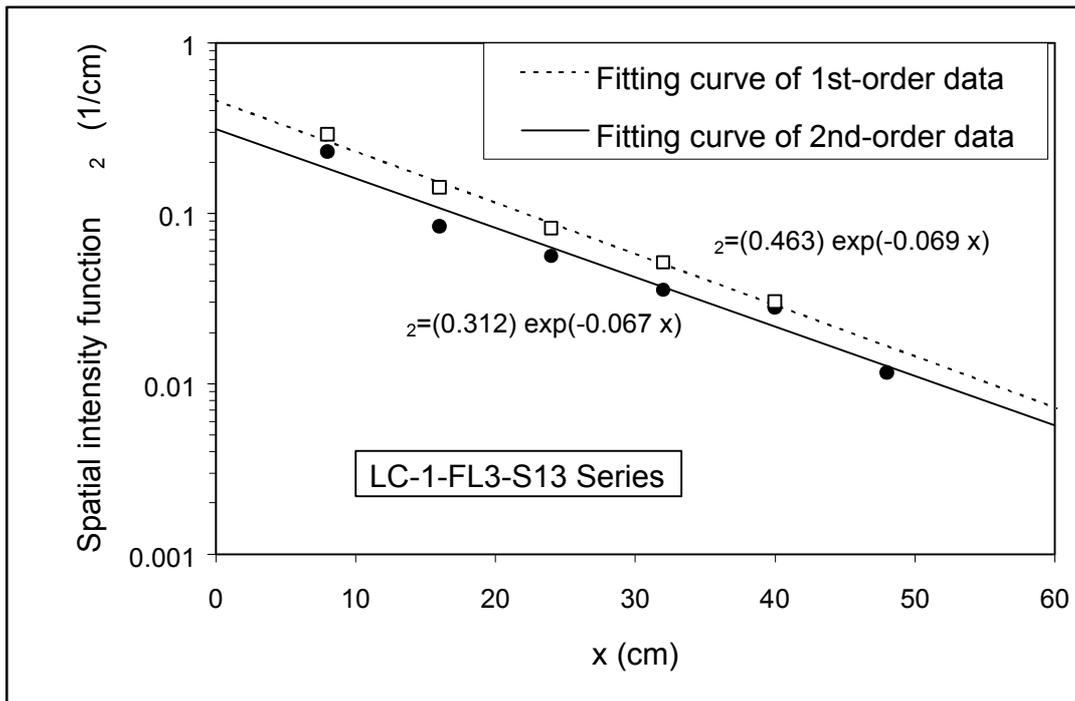


Figure 2: Calculated data and fitting curves of spatial intensity function I_2 (□ 1st-order data, • 2nd-order data)

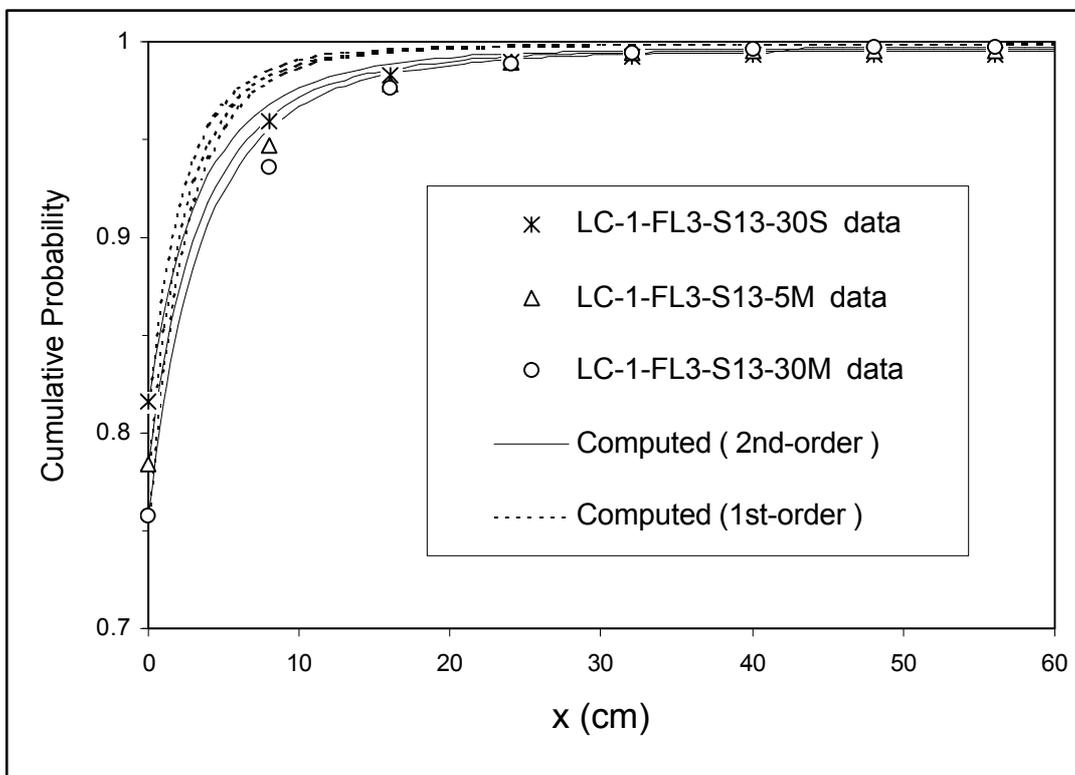


Figure 3: Experimental data and computed cumulative probability distributions using 1st- and 2nd-order approximated parameters

5 Error Analysis

The overall error of the estimated parameter or computed probability, in essence, contains two components. The first portion originates from the underlying assumptions of the non-homogeneous Poisson model employed. The second portion arises from the simplifications in the approximation method. This work mainly investigates the computation error resulting from the use of an approximate parameter that is solved from a system of simplified equations.

5.1 Error of Spatial Intensity Function

The “error” is defined as the difference between the exact magnitude of a quantity (e.g. physical data) and its approximate value (e.g. simulated or computed result). To evaluate the magnitude of error of the spatial intensity function estimated from a system of simplified equations, the error of such system must be considered at first. The error of the simplified CDF listed in Table 1 is, theoretically, the difference between the close and simplified forms of the equation. Obviously, we do not have the exact form of $F_t(x + \Delta x)/F_t(x)$. However, the leading error term of the first-order simplified CDF, $E_{CDF}^{(1)}$, can be estimated by its difference to the second-order simplified CDF, i.e.

$$E_{CDF}^{(1)} = \left[\frac{F_t(x + \Delta x)}{F_t(x)} \right]^{(2)-(1)} = O(\Delta x^2) \quad (4)$$

The order of magnitude analysis reveals a fact that the error of first-order simplified CDF is dominated by a leading term of $O(\Delta x^2)$. It has been proved that an unknown solved from a system with an error of $O(\Delta x^2)$ inherits an error of the same order [4], i.e.

$$a_1^{(1)} = \hat{a}_1 + O(\Delta x^2) \quad (5)$$

in which $a_1^{(1)}$ is the magnitude solved from first-order simplified equations, \hat{a}_1 is the close form solution. Given $\lambda_2 = a_1 / \Delta x$ and (5), the magnitude of the error associated with the spatial intensity function estimated from a system of equations simplified with first-order approximation method can be evaluated [4]:

$$e^{(1)} = \hat{\lambda}_2 - \lambda_2^{(1)} = O(\Delta x) \quad (6)$$

in which $\hat{\lambda}_2$ and $\lambda_2^{(1)}$ are respectively the magnitudes of the exact and first-order approximated spatial intensity function. Similarly, for the i th-order approximation method, the following relations also hold [4]:

$$\begin{aligned} E_{CDF}^{(i)} &= O(\Delta x^{i+1}) \\ e^{(i)} &= O(\Delta x^i) \end{aligned} \quad (7)$$

where $E_{CDF}^{(i)}$ and $e^{(i)}$ are respectively the leading error of the i th-order simplified CDF and error of the spatial intensity function estimated with the i th-order approximation method.

5.2 Error of Cumulative Probability Distribution

The error of the cumulative probability distribution computed with (1) originates from using an approximate spatial intensity function that is solved from a system of simplified equations. The computation error of the cumulative probability gained through the use of first-order approximated spatial intensity function is shown to be on the first order of the finite increment Δx [4], i.e.

$$E^{(1)} = \hat{F}_t(x) - F_t^{(1)}(x) = O(\Delta x) \quad (8)$$

in which $E^{(1)}$ is the error of first-order cumulative probability, $\hat{F}_t(x)$ and $F_t^{(1)}(x)$ are the exact and first-order approximated magnitudes of the cumulative probability at location x . The general form representing the magnitude of computation error for i th-order approximation method can be expressed as the following [4]:

$$E^{(i)} = \hat{F}_t(x) - F_t^{(i)}(x) = O(\Delta x^i) \quad (9)$$

where $E^{(i)}$ is the error of the i th-order approximated magnitude of the cumulative probability at location x , $F_t^{(i)}(x)$.

6 Conclusions

- (1) The stochastic parameters of the NHP model can be estimated by the proposed approximation technique. The fitting curves of the spatial intensity function can be represented by a general form of Eq. (3), in which the exponent b is nearly identical for both first- and second-order curves, but the coefficient a is greater for first-order curves.
- (2) The cumulative probability curves obtained by use of the second-order I_2 curves explicitly reveal better coincidences with the physical data. The second-order approximation technique reduces nearly 50% of the average gross error associated with the first-order cumulative probability distributions.
- (3) The simplified form of the i th-order expanded CDF equation has an error of $O(\Delta x^{i+1})$. The spatial intensity function solved through a system of such equations inherits an error of $O(\Delta x^i)$. The computation error of the cumulative probability distribution gained by use of the i th-order approximated spatial intensity function is in a magnitude of $O(\Delta x^i)$.
- (4) The order of magnitude analysis reveals that the error associated with the proposed approximation method is dependent on the size of the finite increment as well as the degree of approximation. The magnitudes of relevant errors are significantly reduced as Δx is becoming infinitesimal or a sufficiently high-order approximation technique is employed.

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