5. Energy Equation

1. Ideal fluid (理想流體)：Ideal fluid is a fluid assumed to be "inviscid" (non-viscous fluid) 非黏性流體

Mathematically： $\mu = 0$

→ 為何要假設為理想流體？

Ans：假設為理想流體即表示流體粒子間，或流體與邊界間無黏滯力作用，故所有加諸在流體上之力，均用以產生加速度，無需克服阻力。

2. Distinctions between various fluids and flows(各種流體與流況之區別)：

①Newtonian fluid $\left(\tau_{yx} = \mu \dfrac{du}{dy}\right)$

Non-Newtonian fluid

②Incompressible fluid $(\rho = \text{const.})$

Compressible fluid

③Steady flow $\left(\dfrac{\partial(\,)}{\partial t} = 0\right)$

Unsteady flow

④Irrotational flow $(\vec{\Omega} = 0)$

Rotational flow

⑤Ideal fluid (Inviscid fluid, Non-viscous fluid) $(\mu = 0)$

Real fluid (Viscid fluid, Viscous fluid)

⑥Laminar flow (層流)

Turbulent flow (紊流)

3. 1-D Euler's Eqn. (along the streamline)

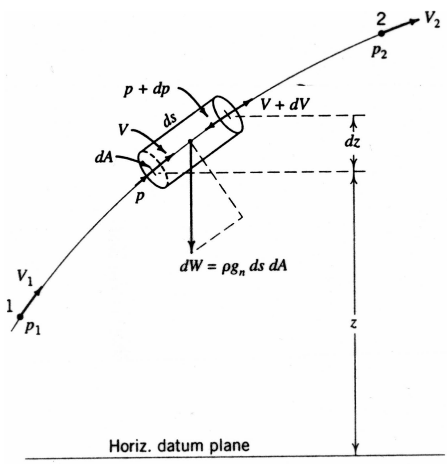


Fig. 5.1

因為是Inviscid fluid ($\mu = 0$) $\rightarrow$ No viscous force(無需考慮黏滯剪應力 τ)

$\rightarrow$ 所受到的力：①Pressure，②Gravity

①Pressure force：

$$P \cdot dA - (P + dP)dA = -dP \cdot dA$$

②Gravity force：

$$dW = \rho g \cdot dA \cdot ds$$

Streamline dir.： $-dW \cdot \sin\theta = -\rho g \cdot dA \cdot ds \cdot \dfrac{dz}{ds}$

$$= -\rho g \cdot dA \cdot dz$$

③ $F_s = m \cdot a_s$

$$F_s = (-dP \cdot dA - \rho g \cdot dA \cdot dz)$$

$$m = \rho \cdot dA \cdot ds$$

$a_s = V \cdot \dfrac{dV}{ds}$ (Recall Chap.3)

$$-dP \cdot dA - \rho g \cdot dA \cdot dz = (\rho \cdot dA \cdot ds)\left(V \cdot \frac{dV}{ds}\right)$$

$$-dP - \rho g \cdot dz = \rho V \cdot dV$$

$$\rightarrow \frac{dP}{\rho} + g \cdot dz + V \cdot dV = 0$$

or $\boxed{\dfrac{dP}{\rho g} + dz + \dfrac{dV^2}{2g} = 0}$ 1-D Euler's eqn. (along the streamline)

For Incompressible fluid: (ρ=const.)

$$d\left(\frac{P}{\rho g} + z + \frac{V^2}{2g}\right) = 0$$

4. Bernoulli's Eqn. along the Streamline

Integrating 1-D Euler's eqn. (along the streamline)

$\boxed{\dfrac{P}{\gamma} + z + \dfrac{V^2}{2g} = \text{constant} = H}$ Bernoulli's eqn. (along the streamline)

◎<u>Assumptions</u>（適用條件）：① $\mu=0$ (Inviscid fluid) 理想流體

② ρ=const. (Incompressible fluid)

③Along the streamline (1-D)

④steady flow

Bernoulli's eqn. is a useful relationship between P ，V ，and z.

where $H\equiv$ Total head (總水頭)

$\frac{P}{\gamma}\equiv$ pressure head (壓力水頭)

$z\equiv$ elevation head (高度水頭)

$\frac{V^2}{2g}\equiv$ velocity head (速度水頭)

5. Energy line (EL) and Hydraulic Grade line (HGL)

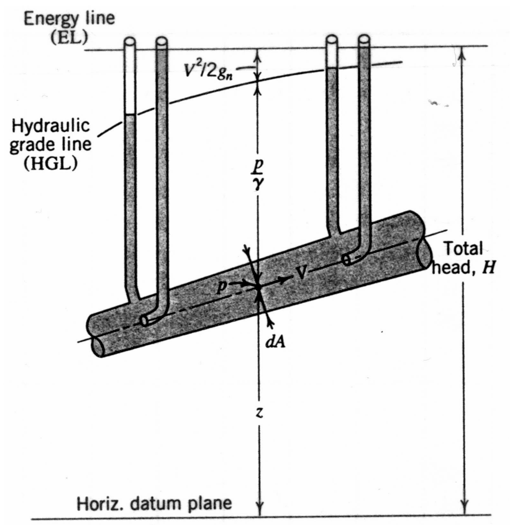


Fig. 5.2

① Energy line (EL)≡ Total head line (可用 Stagnation Tube量測)

$\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)$ 總和之連線

②Hydraulic grade line (HGL) ≡ Piezometric head line (可用 Piezometer量測)

$\left(\frac{P}{\gamma}+z\right)$ 總和之連線 (稱為piezometric head or hydraulic head)

6. Work-Energy Equation

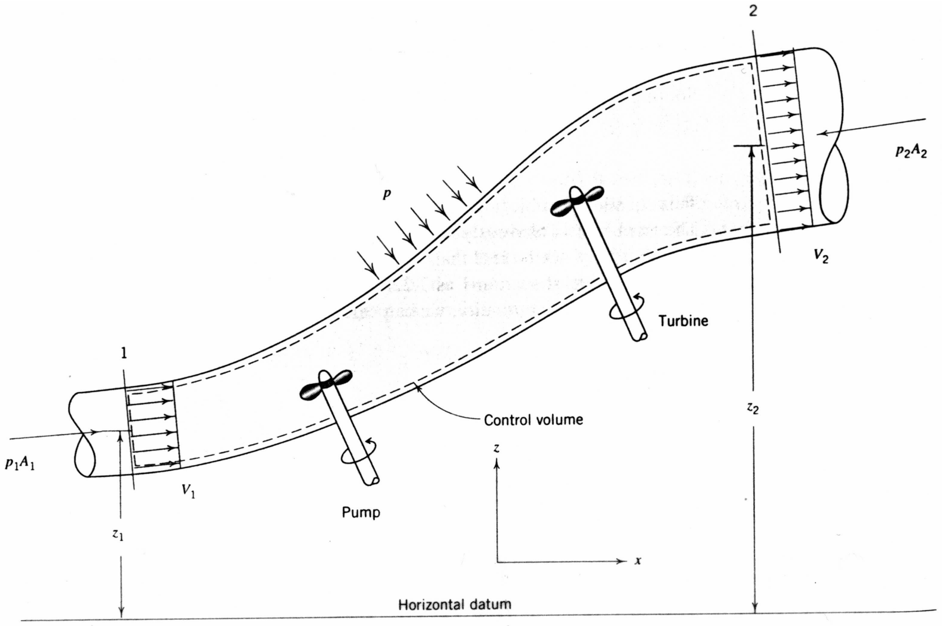


Fig. 5.8

●Mechanical work-energy principle:

(Heat transfer and internal energy are neglected.)

$$dW = dE$$

where $dW =$ the work done on a fluid system

$dE =$ the change in P.E. and K.E. of a fluid system

→ $\dfrac{dW}{dt} = \dfrac{dE}{dt}$

①For steady flow and incompressible fluid:

$$\frac{dE}{dt} = \oiint_{c.s.} e\rho \, \vec{V} \cdot d\vec{A}$$

where $e = \dfrac{E}{m} = \dfrac{\frac{1}{2}mV^2 + mgz}{m} = \dfrac{V^2}{2} + gz$

$$\frac{dE}{dt} = \oiint_{c.s.} \rho \left(\frac{V^2}{2} + gz \right) \vec{V} \cdot d\vec{A}$$

$$= \rho \left(\frac{V_2^2}{2} + gz_2 \right) V_2 A_2 - \rho \left(\frac{V_1^2}{2} + gz_1 \right) V_1 A_1$$

$$= Q\gamma \left[\left(\frac{V_2^2}{2g} + z_2 \right) - \left(\frac{V_1^2}{2g} + z_1 \right) \right]$$

② $\dfrac{dW}{dt}$ ：Two parts

(1) Surface forces: <i> Normal: pressure (✓)

<ii> Shear: friction (✗)

$$\frac{dW}{dt} = \oiint_{c.s.} (-pd\vec{A}) \cdot \vec{V} \qquad (Power = \vec{F} \cdot \vec{V})$$

$$= P_1 A_1 V_1 - P_2 A_2 V_2$$

$$= Q(P_1 - P_2)$$

(2) Mechine work: Turbine $\dfrac{dW}{dt} = -Q\gamma\ E_T$

Pump $\dfrac{dW}{dt} = +Q\gamma\ E_P$

Where E_T = energy extracted by turbine per unit weight of fluid

E_P = energy added by pump per unit weight of fluid

$$\Rightarrow \quad \frac{dW}{dt} = \frac{dE}{dt}$$

$$= Q(P_1 - P_2) + (Q\gamma\ E_P - Q\gamma\ E_T)$$

$$= Q\gamma \left(\frac{P_1}{\gamma} - \frac{P_2}{\gamma} \right) + Q\gamma \left(E_P - E_T \right)$$

$$= Q\gamma \left(\frac{P_1}{\gamma} - \frac{P_2}{\gamma} + E_P - E_T \right)$$

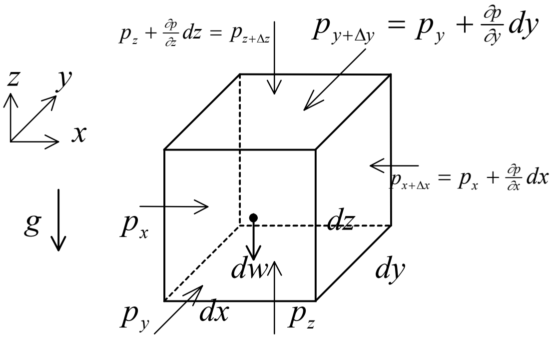
$$= Q\gamma \left[\left(\frac{V_2^2}{2g} + z_2 \right) - \left(\frac{V_1^2}{2g} + z_1 \right) \right]$$

$$\therefore \quad \boxed{\frac{P_1}{\gamma} + z_1 + \frac{V_1^2}{2g} + E_P = \frac{P_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + E_T} \quad \text{Work-Energy Equation}$$

<u>Power</u>: $\dfrac{Q\gamma\ (E_T \text{ or } E_P) \dfrac{ft \cdot lb}{s}}{550} = hp$ (U.S. units)

$\dfrac{Q\gamma\ (E_T \text{ or } E_P) \dfrac{J}{s}}{1000} = kW$ (SI units)

7. 3-D Euler's Equation



Without viscosity (No shear)

→ Only Pressure (normal stress) and gravity (body force)

$$① \sum F_x = P_x\,dy\,dz - \left(P_x + \frac{\partial P}{\partial x}dx\right)dy\,dz = -\frac{\partial P}{\partial x}dx\,dy\,dz$$

$$② \sum F_y = P_y\,dx\,dz - \left(P_y + \frac{\partial P}{\partial y}dy\right)dx\,dz = -\frac{\partial P}{\partial y}dx\,dy\,dz$$

$$③ \sum F_z = P_z dxdy - \left(P_z + \frac{\partial P}{\partial z}dz\right)dxdy - \rho\, dxdydz \cdot g = -\frac{\partial P}{\partial z}dxdydz - \rho\, gdxdydz$$

We know：

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = \frac{Du}{Dt}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = \frac{Dv}{Dt}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = \frac{Dw}{Dt}$$

$$\vec{F} = m\vec{a} = (\rho dxdydz)\vec{a}$$

$$① F_x = ma_x = (\rho\,dx\,dy\,dz)a_x = -\frac{\partial P}{\partial x}dx\,dy\,dz$$

$$\therefore\ \rho a_x = \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right] = -\frac{\partial P}{\partial x}$$

$$② F_y = ma_y = (\rho\, dxdydz)a_y = -\frac{\partial P}{\partial y}dxdydz$$

$$\therefore\ \rho a_y = \rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right] = -\frac{\partial P}{\partial y}$$

③ $F_z = ma_z = (\rho dxdydz)a_z = -\frac{\partial P}{\partial z}dxdydz - \rho g dxdydz$

$\therefore\ \rho a_z = -\frac{\partial P}{\partial z} - \rho g = \rho\left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right]$

整理如下：

$$\rho a_x = \rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x$$

$$\rho a_y = \rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y$$

$$\rho a_z = \rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho g_z = -\frac{\partial P}{\partial z} - \rho g$$

如令 $\vec{g} = g_x\hat{i} + g_y\hat{j} + g_z\hat{k}$　　　　則 $g_x = g_y = 0$，$g_z = -g$

故Euler's Eqn. 通式 $\boxed{\rho\frac{D\vec{V}}{Dt} = -\nabla P + \rho\vec{g}}$ Eqn. of Motion (or Momentum eqn.) for ideal and incompressible fluid

$\Rightarrow$ can slove　4 unknowns：u、v、w、P

by　3 eqns. of motion (x、y、z 方向)

and　1 continuity eqn. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

8. Bernoulli's Eqn. in flowfield

Integrating Euler's eqn. for incompressible fluid and steady flow

$\times dx$　① $-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

$\times dy$　② $-\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$

$\times dz$　③ $-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$

$$\sum \Rightarrow -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) = \left(u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx\right)$$

$$+\left(u\frac{\partial v}{\partial x}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dy\right)$$

$$+\left(u\frac{\partial w}{\partial x}dz + v\frac{\partial w}{\partial y}dz + w\frac{\partial w}{\partial z}dz\right) + gdz$$

$$\Rightarrow -\frac{1}{\rho}(dP)=\left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial x}dx+\frac{\partial\left(\frac{v^2}{2}\right)}{\partial x}dx+\frac{\partial\left(\frac{w^2}{2}\right)}{\partial x}dx\right]$$

$$+\left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial y}dy+\frac{\partial\left(\frac{v^2}{2}\right)}{\partial y}dy+\frac{\partial\left(\frac{w^2}{2}\right)}{\partial y}dy\right]$$

$$+\left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial z}dz+\frac{\partial\left(\frac{v^2}{2}\right)}{\partial z}dz+\frac{\partial\left(\frac{w^2}{2}\right)}{\partial z}dz\right]$$

$$\left[-v\frac{\partial v}{\partial x}dx-w\frac{\partial w}{\partial x}dx-u\frac{\partial u}{\partial y}dy-w\frac{\partial w}{\partial y}dy-u\frac{\partial u}{\partial z}dz-v\frac{\partial v}{\partial z}dz\right]$$

$$+\left(v\frac{\partial u}{\partial y}dx+u\frac{\partial v}{\partial x}dy\right)+\left(w\frac{\partial u}{\partial z}dx+u\frac{\partial w}{\partial x}dz\right)+\left(w\frac{\partial v}{\partial z}dy+v\frac{\partial w}{\partial y}dz\right)+g\,dz$$

$$=d\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)+\left[\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)(u\,dy-v\,dx)\right]+\left[\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)(v\,dz-w\,dy)\right]$$

$$+\left[\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)(w\,dx-u\,dz)\right]+g\,dz$$

$$\left(\times\frac{1}{g}\right)\Rightarrow -\frac{dP}{\gamma}=\frac{d\left(u^2+v^2+w^2\right)}{2g}+\frac{1}{g}\left[\Omega_x(v\,dz-w\,dy)+\Omega_y(w\,dx-u\,dz)+\Omega_z(u\,dy-v\,dx)\right]+dz$$

⇒Integration：

$$\boxed{\frac{P}{\gamma}+\frac{V^2}{2g}+z=H-\frac{1}{g}\int\left[\Omega_x(v\,dz-w\,dy)+\Omega_y(w\,dx-u\,dz)+\Omega_z(u\,dy-v\,dx)\right]}$$

●Bernoulli's eqn. (any two points in flowfield) valid for ① $\mu=0$ (Euler's eqn.)

② ρ=const

③steady flow

④irrotational ($\vec{\Omega}=0$)

- If rotational，only valid when $\left(v\,dz - w\,dy\right) = \left(w\,dx - u\,dz\right) = \left(u\,dy - v\,dx\right) = 0$

$\Rightarrow \dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$ (streamline function)

$\Rightarrow$Bernoulli's eqn. only valid along streamline in rotational flows.

5. Energy Equation for Ideal Fluid

1. Ideal fluid (理想流體)： **Ideal fluid is a fluid assumed to be "inviscid" (i.e., non-viscous)** 非黏性流體

Mathematically：$\mu = 0$

→ **Question:** 為何要假設為理想流體？

Answer:

假設為理想流體即表示流體粒子間、或流體與邊界間無黏滯力作用，故所有加諸在流體上之力，均用以產生加速度，無需克服阻力，亦無能量損失。

1

2. 1-D Euler's Eqn. (along the streamline)

Euler (Swiss, 1707-1783)

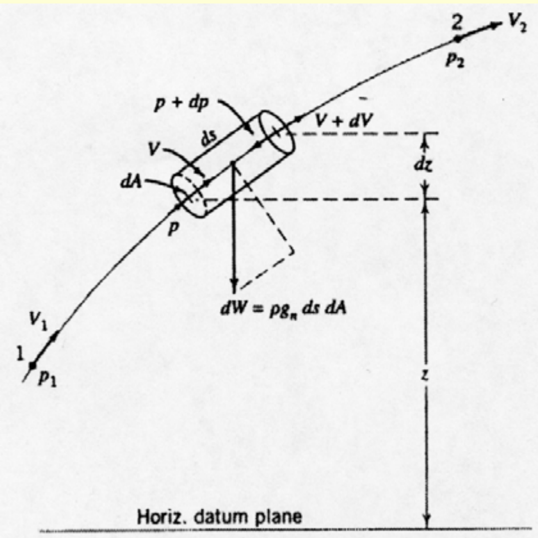


Fig. 5.1

2

因為是Inviscid fluid ($\mu = 0$) → No viscous force(無需考慮黏滯剪應力 τ)

→ 所受到的力：①Pressure，②Gravity

① Pressure force：

$$P \cdot dA - (P + dP)dA = -dP \cdot dA$$

② Gravity force：

$$dW = \rho g \cdot dA \cdot ds$$

Streamline dir.： $-dW \cdot \sin\theta = -\rho\, g \cdot dA \cdot ds \cdot \dfrac{dz}{ds}$

$$= -\rho\, g \cdot dA \cdot dz$$

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③ $F_s = m \cdot a_s$

$$F_s = (-dP \cdot dA - \rho\, g \cdot dA \cdot dz)$$

$$m = \rho \cdot dA \cdot ds$$

$a_s = V \cdot \dfrac{dV}{ds}$ (Recall Chap.3)

$$-dP \cdot dA - \rho\, g \cdot dA \cdot dz = (\rho \cdot dA \cdot ds)\left(V \cdot \frac{dV}{ds}\right)$$

$$-dP - \rho g \cdot dz = \rho V \cdot dV$$

$$\rightarrow \frac{dP}{\rho} + g \cdot dz + V \cdot dV = 0$$

or $\boxed{\dfrac{d\,P}{\rho\, g} + dz + \dfrac{dV^2}{2g} = 0}$ 1-D Euler's eqn. (along the streamline)

For incompressible fluid: (ρ =const.)

$$d\left(\frac{P}{\rho\, g} + z + \frac{V^2}{2g}\right) = 0$$

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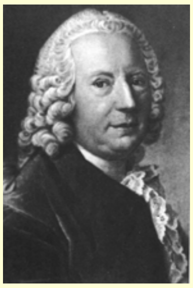
3. Bernoulli's Eqn. along the Streamline



Bernoulli (Swiss, 1700-1782)

Integrating 1-D Euler's eqn. (along the streamline)

$$\boxed{\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant} = H}$$

Bernoulli's eqn. (along the streamline)

◎ **Assumptions (適用條件)**：① $\mu = 0$ **(Inviscid fluid)** 理想流體

② $\rho = \text{const.}$ **(Incompressible fluid)**

③ **Along the streamline (1-D)**

④ **steady flow**

5

Bernoulli's eqn. is a useful relationship between *P*，*V*，and *z*

where $H \equiv$ **Total head (總水頭)**

$\frac{P}{\gamma} \equiv$ **pressure head (壓力水頭)**

$z \equiv$ **elevation head (高度水頭)**

$\frac{V^2}{2g} \equiv$ **velocity head (速度水頭)**

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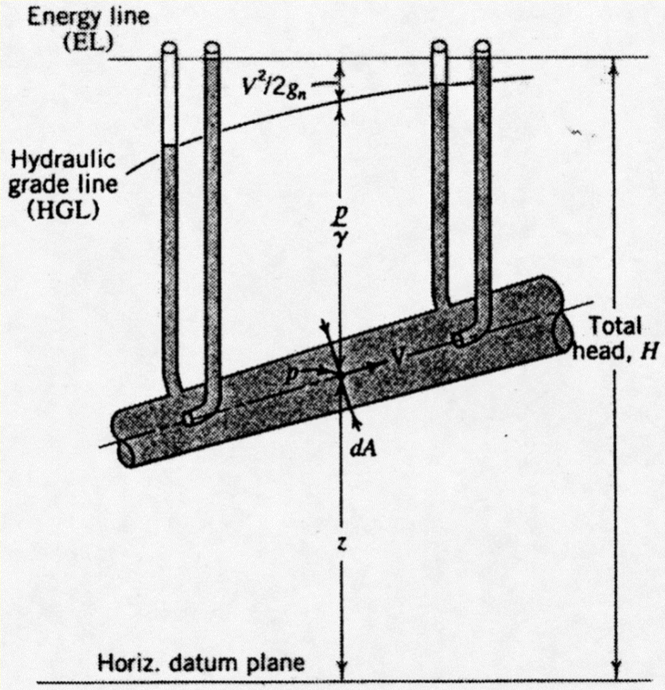
4. Energy Line (EL，能量線) and Hydraulic Grade Line (HGL，水力坡降線)



① **Energy line (EL) ≡ Total head line (可用Stagnation Tube量測)**

$$\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)$$ 總和之連線

② **Hydraulic grade line (HGL) ≡ Piezometric head line**

(可用Piezometer量測)

$$\left(\frac{P}{\gamma}+z\right)$$ 總和之連線 **(稱為piezometric head or hydraulic head)**

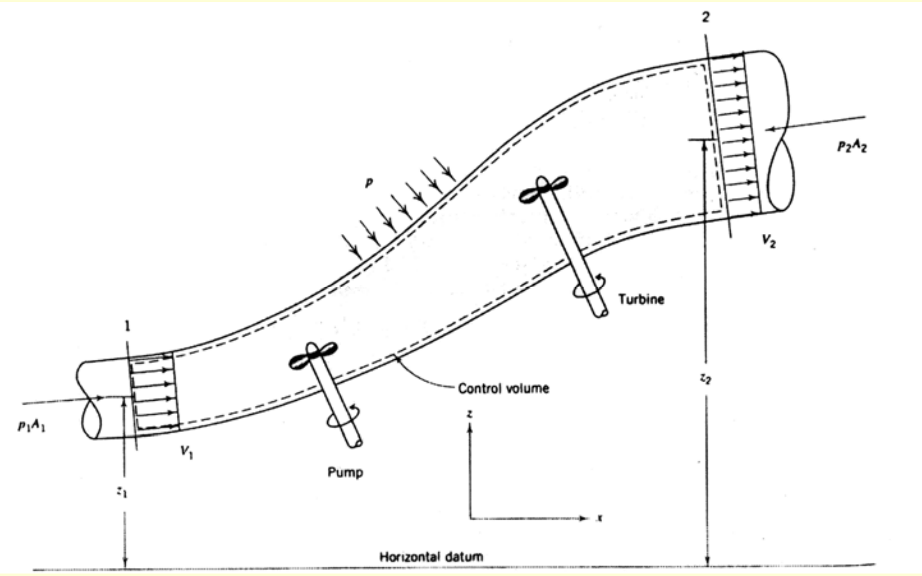
Applications of Bernoulli's Equation (BE)

Type 1: Manometer (壓力計)
BE

Type 2: Manometer (壓力計)
CE
BE

Type 3: CE
BE

5. Work-Energy Equation (功-能方程式)



- **Mechanical work-energy principle:**

(Heat transfer and internal energy are neglected.)

$$dW = dE$$

where dW = the work done on a fluid system

dE = the change in P.E. and K.E. of a fluid system

$$\rightarrow \frac{dW}{dt} = \frac{dE}{dt}$$

① For steady flow and incompressible fluid:

$$\frac{dE}{dt} = \oiint_{c.s.} e\rho \vec{V} \cdot d\vec{A}$$

where $e = \dfrac{E}{m} = \dfrac{\frac{1}{2}mV^2 + mgz}{m} = \dfrac{V^2}{2} + gz$

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$$\frac{dE}{dt} = \oiint_{c.s.} \rho \left(\frac{V^2}{2} + gz \right) \vec{V} \cdot d\vec{A}$$

$$= \rho \left(\frac{V_2^2}{2} + gz_2 \right) V_2 A_2 - \rho \left(\frac{V_1^2}{2} + gz_1 \right) V_1 A_1$$

$$= Q\gamma \left[\left(\frac{V_2^2}{2g} + z_2 \right) - \left(\frac{V_1^2}{2g} + z_1 \right) \right]$$

② $\dfrac{dW}{dt}$: Two parts

(1) Surface forces: <i> Normal: pressure (✓)

<ii> Shear: friction (✗)

$$\frac{dW}{dt} = \oiint_{c.s.} (-pd\vec{A}) \cdot \vec{V} \qquad (Power = \vec{F} \cdot \vec{V})$$

$$= P_1 A_1 V_1 - P_2 A_2 V_2$$

$$= Q(P_1 - P_2)$$

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(2) Mechine work: Turbine $\frac{dW}{dt} = -Q\gamma \, E_T$

Pump $\frac{dW}{dt} = +Q\gamma \, E_P$

where E_T = energy extracted by turbine per unit weight of fluid

E_P = energy added by pump per unit weight of fluid

$$\Rightarrow \quad \frac{dW}{dt} = \frac{dE}{dt}$$

$$= Q(P_1 - P_2) + (Q\gamma \, E_P - Q\gamma \, E_T)$$

$$= Q\gamma\left(\frac{P_1}{\gamma} - \frac{P_2}{\gamma}\right) + Q\gamma\left(E_P - E_T\right)$$

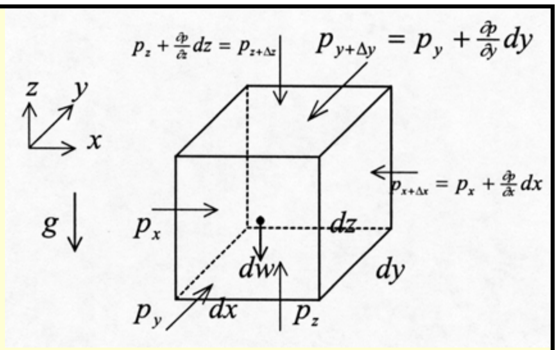
$$= Q\gamma\left(\frac{P_1}{\gamma} - \frac{P_2}{\gamma} + E_P - E_T\right)$$

$$= Q\gamma\left[\left(\frac{V_2^2}{2g} + z_2\right) - \left(\frac{V_1^2}{2g} + z_1\right)\right]$$

$$\therefore \quad \boxed{\frac{P_1}{\gamma} + z_1 + \frac{V_1^2}{2g} + E_P = \frac{P_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + E_T}$$ Work-Energy Equation

<u>Power:</u> $\frac{Q\gamma \, (E_T \text{ or } E_P) \frac{J}{s}}{1000} = kW$ (kilo watt)

6. 3-D Euler's Equation in Flowfield



Without viscosity (No shear force)

→ Only Pressure (normal stress) and gravity (body force)

① $$\sum F_x = P_x\,dy\,dz - \left(P_x + \frac{\partial P}{\partial x}dx\right)dy\,dz = -\frac{\partial P}{\partial x}dx\,dy\,dz$$

② $$\sum F_y = P_y\,dx\,dz - \left(P_y + \frac{\partial P}{\partial y}dy\right)dx\,dz = -\frac{\partial P}{\partial y}dx\,dy\,dz$$

③ $$\sum F_z = P_z dxdy - \left(P_z + \frac{\partial P}{\partial z}dz\right)dxdy - \rho\,dxdydz \cdot g$$

$$= -\frac{\partial P}{\partial z}dxdydz - \rho\,gdxdydz$$

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We know : $$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = \frac{Du}{Dt}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = \frac{Dv}{Dt}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = \frac{Dw}{Dt}$$

$$\vec{F} = m\vec{a} = (\rho\,dx\,dy\,dz)\vec{a}$$

① $$F_x = ma_x = (\rho\,dx\,dy\,dz)a_x = -\frac{\partial P}{\partial x}dx\,dy\,dz$$

$$\therefore\quad \rho a_x = \rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right] = -\frac{\partial P}{\partial x}$$

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② $F_y = ma_y = (\rho dxdydz)a_y = -\frac{\partial P}{\partial y}dxdydz$

$$\therefore \quad \rho a_y = \rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}\right] = -\frac{\partial P}{\partial y}$$

③ $F_z = ma_z = (\rho dxdydz)a_z = -\frac{\partial P}{\partial z}dxdydz - \rho g dxdydz$

$$\therefore \quad \rho a_z = -\frac{\partial P}{\partial z} - \rho g = \rho\left[u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}\right]$$

整理如下： $\rho a_x = \rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x$

$$\rho a_y = \rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y$$

$$\rho a_z = \rho\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \rho g_z = -\frac{\partial P}{\partial z} - \rho g$$

令 $\bar{g} = g_x\hat{i} + g_y\hat{j} + g_z\hat{k}$ 　　則 $g_x = g_y = 0$，$g_z = -g$

故Euler's Eqn. 通式 $\boxed{\rho\frac{D\bar{V}}{Dt} = -\nabla P + \rho\bar{g}}$

⇨ **It can be solved!** 　**4 unknowns**：u、v、w、P

by 　**3 eqns. of motion** (x、y、z 方向)

and 　**1 continuity eqn.** $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

3-D Euler's Eqn. is an equation of motion for ideal and incompressible fluid.

In Chapter 9, we will talk about the equation of motion for real fluids, which is Navier-Stokes Equation.

7. Bernoulli's Eqn. in Flowfield

Integrating 3-D Euler's eqn. for <u>incompressible fluid</u> and <u>steady flow</u>

$$\times dx \quad ① \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\times dy \quad ② \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\times dz \quad ③ \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} - g = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\sum \Rightarrow -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)$$

$$= \left(u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx\right) + \left(u\frac{\partial v}{\partial x}dy + v\frac{\partial v}{\partial y}dy + w\frac{\partial v}{\partial z}dy\right)$$

$$+ \left(u\frac{\partial w}{\partial x}dz + v\frac{\partial w}{\partial y}dz + w\frac{\partial w}{\partial z}dz\right) + g\,dz$$

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$$\Rightarrow -\frac{1}{\rho}(dP) = \left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial x}dx + \frac{\partial\left(\frac{v^2}{2}\right)}{\partial x}dx + \frac{\partial\left(\frac{w^2}{2}\right)}{\partial x}dx\right]$$

$$+ \left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial y}dy + \frac{\partial\left(\frac{v^2}{2}\right)}{\partial y}dy + \frac{\partial\left(\frac{w^2}{2}\right)}{\partial y}dy\right]$$

$$+ \left[\frac{\partial\left(\frac{u^2}{2}\right)}{\partial z}dz + \frac{\partial\left(\frac{v^2}{2}\right)}{\partial z}dz + \frac{\partial\left(\frac{w^2}{2}\right)}{\partial z}dz\right]$$

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$$\left[-v\frac{\partial v}{\partial x}dx-w\frac{\partial w}{\partial x}dx-u\frac{\partial u}{\partial y}dy-w\frac{\partial w}{\partial y}dy-u\frac{\partial u}{\partial z}dz-v\frac{\partial v}{\partial z}dz\right]$$

$$+\left(v\frac{\partial u}{\partial y}dx+u\frac{\partial v}{\partial x}dy\right)+\left(w\frac{\partial u}{\partial z}dx+u\frac{\partial w}{\partial x}dz\right)+\left(w\frac{\partial v}{\partial z}dy+v\frac{\partial w}{\partial y}dz\right)+gdz$$

$$=d\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)+\left[\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)(udy-vdx)\right]+\left[\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)(vdz-wdy)\right]$$

$$+\left[\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)(wdx-udz)\right]+gdz$$

$$\left(\times\frac{1}{g}\right)\Rightarrow-\frac{dP}{\gamma}=\frac{d\left(u^2+v^2+w^2\right)}{2g}$$

$$+\frac{1}{g}\left[\Omega_x(vdz-wdy)+\Omega_y(wdx-udz)+\Omega_z(udy-vdx)\right]+dz$$

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⇨ **Integration：**

$$\boxed{\frac{P}{\gamma}+\frac{V^2}{2g}+z=H-\frac{1}{g}\int\left[\Omega_x(vdz-wdy)+\Omega_y(wdx-udz)+\Omega_z(udy-vdx)\right]}$$

- **Bernoulli's eqn. (any two points in flowfield) valid for** ① $\mu=0$ **(Euler's eqn.)**
  ② $\rho=$ **const**
  ③ **steady flow**
  ④ **irrotational** $(\vec{\Omega}=0)$

- **If rotational，only valid when** $(vdz-wdy)=(wdx-udz)=(udy-vdx)=0$

$$\Rightarrow\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}\quad\text{(streamline equation)}$$

⇒ **Bernoulli's eqn. only valid along streamline in rotational flows.**

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